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# Infinite Friendship Graphs with Infinite Parameters*

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We study infinite graphs in which every set of $\kappa$ vertices has exactly $\lambda$ common neighbours. We prove that there exist $2^\sigma$ such graphs of each infinite order $\sigma$ if $\kappa$ is finite and that for $\kappa$ infinite there are $2^\lambda$ graphs of order $\lambda$ and none of cardinality greater than $\lambda$ (assuming the GCH). Further, we show that all a priori admissible chromatic numbers are in fact possible for such graphs. © 1991 Academic Press, Inc.

## 1. PRELIMINARIES

Let $\kappa$ and $\lambda$ be cardinals, finite or infinite. By a (generalized) *friendship graph* we mean a (simple, undirected) graph with the property that every set of $\kappa$ vertices has *exactly* $\lambda$ common neighbours. In order to avoid trivialities we assume that a friendship graph has at least $\kappa$ vertices and that $\kappa \geqslant 2$. The class of friendship graphs with parameters $\kappa$ and $\lambda$ is denoted by $\mathscr{G}_k^\lambda$ and the subclass consisting of the infinite ones by $\mathscr{I}_\kappa^\lambda$. The idea and the name are old—Erdős *et al.* [7] proved their *Friendship Theorem* in 1966; the name was coined by Wilf some time later.

Let us first settle on some notation. A graph $G$ here is a pair $(V(G), E(G))$ with the edge set $E(G)$ being a subset of the set of two-element subsets of the vertex set $V(G)$. For the most part we abuse notation and write $x \in G$ for $x \in V(G)$ and $S \cap G$ for $S \cap V(G)$. We also write $S \subset G$ to indicate $S \subset V(G)$. For $x \in G$ we write $N(x)$ for the neighbourhood of $x$,

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i.e., $N(x) = \{y \in G\colon \{x, y\} \in E(G)\}$. The set of common neighbours of a set $T \subset G$ is the set $N(T) = \bigcap_{x \in T} N(x)$. Cardinals are considered as the least ordinals of a given power.

The study of friendship graphs naturally separates into four parts:

1. $\kappa$ and $\lambda$ finite
   - $\mathscr{G}_\kappa^\lambda - \mathscr{I}_\kappa^\lambda$
   - $\mathscr{I}_\kappa^\lambda$
2. $\lambda$ infinite
   - $\kappa$ finite
   - $\kappa$ infinite.

## 2. Friendship Graphs with $\kappa$ and $\lambda$ Finite

### 2.1. *Finite Graphs in $\mathscr{G}_\kappa^\lambda$ for Both Parameters Finite*

With the exception of $\kappa = 2$ and $\lambda > 1$, this case is closed. When $\kappa = 2$ and $\lambda = 1$, the Friendship Theorem of Erdős, Rényi, and Sós [7] says that there is exactly one friendship graph for any odd order (none of even) and that this graph contains a vertex adjacent to all others. For $\kappa > 2$, the only friendship graph is the complete graph of order $\kappa + \lambda$ [3, 10]. For $\kappa = 2$ and $\lambda > 1$, the finite graphs in $\mathscr{G}_\kappa^\lambda$ are regular ([8] and Erdős quoted in [9]) and only a few examples are known (constructed mostly by Doyen). For a more detailed survey of these result and of other generalizations of the original friendship graph idea see Bondy [2] and Delorme and Hahn [5].

### 2.2. *The Class $\mathscr{I}_\kappa^\lambda$ for Both Parameters Finite*

The paper by Delorme and Hahn essentially completes the study of this case. They obtain the following theorems, the second of which generalizes a similar result of Chvátal, Davies, Kotzig, and Rosenberg [4].

Theorem 2.2.1. *The class $\mathscr{I}_\kappa^\lambda$ is empty if and only if $\kappa > \lambda + 1$.*

Theorem 2.2.2. *When $\mathscr{I}_\kappa^\lambda$ is non-empty, it contains $2^\sigma$ non-isomorphic graphs of order $\sigma$ and chromatic number $d$ for each infinite cardinal $\sigma$ and each $d$ such that $\kappa + 1 \leqslant d \leqslant \aleph_0$.*

Theorem 2.2.3. *If $\lambda \geqslant 2$ and $G \in \mathscr{I}_\kappa^\lambda$ is of order $\sigma$ then both $G$ and its complement are regular of degree $\sigma$.*

THEOREM 2.2.4. *Each vertex of a graph $G \in \mathcal{I}_\kappa^\lambda$ of order $\sigma$ lies in a clique of order $\kappa - 1$ whose (common) neighborhood has cardinality $\sigma$.*

Some of these results and their proof techniques are useful in the subsequent sections.

## 3. FRIENDSHIP GRAPHS WITH $\lambda$ INFINITE

Clearly in this case $\mathcal{G}_\kappa^\lambda = \mathcal{I}_\kappa^\lambda$ and we restrict our notation to $\mathcal{G}_\kappa^\lambda$. Further, if $\mathcal{G}_\kappa^\lambda$ is non-empty, then $\kappa$ is strictly smaller than $\lambda$. To see this, observe that in any graph in $\mathcal{G}_\kappa^\lambda$, the union $S$ of a $\kappa$-element set with the set of its common neighbours cannot have cardinality equal to $\kappa$ since the neighbourhood of $S$ is empty. The last theorem of the preceeding section can be extended as follows.

LEMMA 3.10.1. *Let $G \in \mathcal{G}_\kappa^\lambda$ and let $S$ be a clique of order $\mu < \kappa^+$, the successor of $\kappa$. Then $S \subset K$ for some clique $K$ of cardinality $\kappa^+$.*

*Proof.* Construct a sequence $\{x_\alpha : \alpha < \kappa^+\}$ so that $x_\alpha$ is in the common neighbourhood of $S \cup \{x_\beta : \beta < \alpha\}$. This is always possible since $\lambda > \kappa$ is infinite; the desired clique is $S \cup \{x_\alpha : \alpha < \kappa^+\}$. ∎

### 3.1. $\kappa$ FINITE

The results of [5] provide the maximum possible number of graphs of each cardinality and each possible chromatic number when both $\kappa$ and $\lambda$ are finite. The upper bound of $\aleph_0$ on the chromatic number in Theorem 2.2.2 comes from Corollary 5.6 of [6]. In the case of $\lambda$ infinite the upper bound given by the corollary is $\lambda$. We give here a direct and different proof of this result of [6].

THEOREM 3.1.1 (Erdős and Hajnal [6]). *Let $\mu$ be an infinite cardinal and let $G$ be a graph with chromatic number $\chi(G) \geqslant \mu^+$. Then the complete bipartite graph $K_{n,\mu^+}$ is a subgraph of $G$ for all finite $n$.*

*Proof.* Fix $n$. The proof is by induction on $|G|$. Suppose that $\chi(G) \geqslant \mu^+$ but $K_{n,\mu^+}$ is not a subgraph of $G$. Fix an enumeration $\{g_\xi\}_{\xi < |G|}$ of $G$. We construct a disjoint family $\{S_\alpha\}$ of subsets of $G$ such that $|\bigcup_{\beta<\alpha} S_\beta| \leqslant |\alpha| + \mu$, and if $g \in G \setminus \bigcup_{\beta<\alpha} S_\beta$ then $g$ is adjacent to fewer than $n$ elements of $\bigcup_{\beta<\alpha} S_\beta$. Assume $S_\beta$, $\beta < \alpha$, to be constructed and let $\xi$ be the least such that $g_\xi \notin \bigcup_{\beta<\alpha} S_\beta$. Set

$$T_0 = \{g_\xi\} \cup \bigcup_{\beta<\alpha} S_\beta,$$

$$T_{k+1} = T_k \cup \bigcup \{N(X) : X \subset T_k, |X| = n\} \qquad (k < \omega),$$

and

$$S_\alpha = \bigcup_{k<\omega} T_k - \bigcup_{\beta<\alpha} S_\beta.$$

Note that, by the induction hypothesis, $|T_0| \leqslant |\alpha| + \mu$, and hence (by induction on $k$) $|T_k| \leqslant |\alpha| + \mu$ $(k<\omega)$, and so $|S_\alpha| \leqslant |\alpha| + \mu$. Further, the construction guarantees that if $g \in G \backslash \bigcup_{\beta \leqslant \alpha} S_\beta$ then $|N(g) \cap \bigcup_{\beta \leqslant \alpha} S_\beta| < n$.

Now $\bigcup_{\alpha<|G|} S_\alpha = G$ and $|S_\alpha| \leqslant |\alpha| + \mu < |G|$, so as $K_{n,\mu^+}$ is not a subgraph of the graph $\langle S_\alpha \rangle$ induced by $S_\alpha$, we must have $\chi(\langle S_\alpha \rangle) \leqslant \mu$ for each $\alpha$. Fix a good $\mu$-colouring $c_\alpha$: $S_\alpha \to \mu$ of $\langle S_\alpha \rangle$. We inductively construct a good colouring $c$ of $G$, using as colours the pairs in $\mu \times \omega$. Suppose that $c$ has been defined on $\bigcup_{\beta<\alpha} S_\beta$. For $s \in S_\alpha$, let $m$ be the least such that $c(x) \neq (c_\alpha(s), m)$ for all $x \in N(s) \cap \bigcup_{\beta<\alpha} S_\beta$ (there is such an $m$ since $|N(s) \cap \bigcup_{\beta<\alpha} S_\beta| < n$), and set $c(s) = (c_\alpha(s), m)$. We note that the restriction of $c$ to $\bigcup_{\beta \leqslant \alpha} S_\beta$ is now a good colouring, for if $s$ is adjacent to $t$ then if $s$, $t \in \bigcup_{\beta<\alpha} S_\beta$, the induction hypothesis applies, if $t \in \bigcup_{\beta<\alpha} S_\beta$, $s \in S_\alpha$, then $c(s) \neq c(t)$ by construction, and if $s$, $t \in S_\alpha$, then $c_\alpha(s) \neq c_\alpha(t)$ and this ensures that $c(s) \neq c(t)$. Therefore $c$ is a good colouring of $G$ with $\mu \cdot \aleph_0 = \mu$ colours, contradicting the hypothesis that $\chi(G) \geqslant \mu^+$. ∎

The proof of the next theorem is a direct and simple adapatation of the analogous result of [5, Lemma 2 and Proposition 6]. We give it here because we wish to draw certain conclusions on the basis of some features of the constructions. Lemma 2 of [5] is essentially a closure observation which depends on the finiteness of $\kappa$; it is reproduced here with slight changes. We also modify the proof of Proposition 6 of [5] so that it can be used whether $\kappa$ is finite or infinite. For consistency of notation in this paper these appear in the following definitions and lemmas.

Definition 3.1.1. Let $G$ be any graph not containing the complete bipartite graphs $K_{\kappa,\lambda^+}$. The closure $G^*$ of $G$ (with respect to $\kappa$ and $\lambda$) is the limit of the sequence $\langle G_i : i<\omega \rangle$ obtained by putting $G_0 = G$ and, for each $0<i<\omega$, constructing $G_i$ as follows. For every set $K$ of $\kappa$ vertices in $G_{i-1}$ having $\mu$ common neighbours, add $\lambda - \mu$ new independent vertices together with the new edges joining each vertex of $K$ to each new neighbour.

Obviously in the case of $\lambda$ infinite this just means adding $\lambda$ new neighbours whenever the set $K$ has fewer than $\lambda$ of them. The definition, however, makes sense even for finite $\lambda$.

The following adapts Lemma 2 of [5] (which, in turn, adapted a result of [4]).

Lemma 3.1.1. *Let $G$ be an infinite graph of order $\sigma$ not containing $K_{\kappa,\lambda^+}$ and having chromatic number $d$, $\kappa+1 \leqslant d \leqslant \lambda^+$. Then $G^*$ is in $\mathscr{G}^\lambda_\kappa$, is of* *order $\sigma$, and has chromatic number $d$. If $H$ is a finite subgraph of $G^*$ of minimum degree at least $\kappa+1$ then $H$ is in fact a subgraph of $G$.*

*Proof.* It is easy to see that $G^*$ is in $\mathscr{G}_\kappa^\lambda$. Since $\kappa<\omega$, any set of $\kappa$ vertices of $G^*$ is contained in each $G_i$ for all $i \geqslant i_0$ and has $\lambda$ common neighbours in every such $G_i$. It is equally easy to see that $\chi(G^*)=\chi(G)$ provided $\chi(G) \geqslant \kappa+1$ since at each stage we add independent vertices, each having degree at most $\kappa$ in $G_i$. Noting that each vertex of $G_i \backslash G_{i-1}$ has degree at most $\kappa$ we see that if $H$ is a finite subgraph of $G^*$ of minimum degree $\kappa+1$, it must lie entirely in $G$. ∎

The next definition, an improvement on the idea of [4], allows the construction of the maximum number of non-isomorphic friendship graphs. Before giving it, we recall that given a positive integer $m$, the graph whose vertex set is $\{0,1\}^m$ and whose edges join pairs of points which differ in precisely one coordinate is called the *m-cube*. It is a well-known fact that an $m$-cube is a bipartite regular graph of degree $m$. We denote by $C_m$ the graph obtained from the $m$-cube by deleting the vertex $(0, 0, ..., 0)$ and all edges incident with it; also, we denote by $u_m$ and $v_m$ two special vertices of $C_m$ of even distance and such that $d(v_m)=m$ and $d(u_m)=m-1$ (for example, we can take $u_m=(1,0,0,...,0)$ and $v_m=(1,1,0,0,...,0)$—their distance is 2 and the degrees are as required).

DEFINITION 3.1.2. Let $n>m>2$ be fixed integers. For an infinite ordinal $\mu$ and a function $F\colon \mu^{(2)} \rightarrow \{m, n\}$ (where $\mu^{(2)}$ is the set of two-element subsets of $\mu$) we denote by $G(\mu, F)$ the ordinal graph obtained from $\mu$ (considered as an independent set) by adding, for each pair $\{\alpha, \beta\}$ $(\alpha<\beta<\mu)$, a copy of the graph $C_k$, where $k=F(\{\alpha, \beta\})$, in which $\alpha$ is identified with $u_k$ and $\beta$ with $v_k$.

Note that since the cubes are bipartite and the distance between the special vertices $u$ and $v$ is even, the ordinal graph $G(\mu, F)$ we have defined has chromatic number 2.

LEMMA 3.1.2. *Let $m$ and $n$ be given, $n>m>2$. If $\mu$ and $\nu$ are ordinals and $F_\mu\colon \mu^{(2)} \rightarrow \{m, n\}$, $F_\nu\colon \nu^{(2)} \rightarrow \{m, n\}$ functions, then the ordinal graphs $G(\mu, F_\mu)$ and $G(\nu, F_\nu)$ are isomorphic if and only if $(\mu, F_\mu)=(\nu, F_\nu)$.*

*Proof.* We show that both $\mu$ and $F$ can be recovered from $G(\mu, F)$. First, observe that the elements of $\mu$ correspond to the vertices of $G(\mu, F)$ of infinite degree. Let $\alpha, \beta$ be two such vertices and let $H_{\alpha,\beta}$ be the subgraph obtained from $G(\mu, F)$ by deleting all vertices of infinite degree other than $\alpha, \beta$. The two-connected component of $H_{\alpha,\beta}$ containing $\alpha$ and $\beta$ is $C_{F(\alpha,\beta)}$ and $\alpha<\beta$ if and only if $d(\alpha)<d(\beta)$ in this component. Since $d(\beta)$ is either $m$ or $n$ in $H_{\alpha,\beta}$, this recovers $F$ as well. ∎

COROLLARY 3.1.1. *There are $2^{|\mu|}$ ordinal graphs with given $m$, $n$, $\mu$.*

*Proof.* Each of the $\mu$ pairs of elements of $\mu$ can be joined by one of $C_m, C_n$. ▮

From the results just described we obtain an extension of Theorem 2.2.2 to $\lambda$ infinite.

THEOREM 3.1.2. *Let $\lambda$ be infinite and $\kappa$ finite and let $\sigma \geqslant \lambda$ and $\kappa < d < \lambda^+$. Then $\mathscr{G}_\kappa^\lambda$ contains $2^\sigma$ pairwise non-isomorphic graphs of order $\sigma$ and chromatic number $d$.*

*Proof.* It is sufficient to produce $2^\sigma$ pairwise non-isomorphic graphs not containing $K_{\kappa,\lambda^+}$ of order $\sigma$ and given chromatic number $d$ and in which each vertex belongs to a finite induced subgraph of minimum degree at least $\kappa+1$—the closure lemma guarantees the rest. But this is easy. Unlike in the case of $\lambda$ finite, the graphs needed come ready-made. To wit, let $n > m > \kappa+1$ and let $G(\mu, F)$ be an ordinal graph. Since it has chromatic number 2, the graph $G_{F,d}$ obtained from it by the addition of a disjoint copy of the complete graph on $d$ vertices, $d \geqslant \kappa+1$, has chromatic number $d$. Since $G(\mu, F)$ can clearly be recovered from $G_{F,d}$, the graphs $G_{F,d}$ and $G_{F',d'}$ are isomorphic if and only if $(F, d) = (F', d')$. ▮

The analogue of Theorem 2.2.3 is not quite true when $\kappa$ is finite and $\lambda$ infinite. That is, while $G \in \mathscr{G}_\kappa^\lambda$ is regular, its complement need not be. For example, the graph $G = K \vee S$, the join of the graphs $K$ and $S$ (all vertices of $K$ are adjacent to all vertices of $S$) obtained from the complete graph $K$ on $\lambda$ vertices and the independent set $S$ of order less than $\lambda$ is in $\mathscr{G}_\kappa^\lambda$ and is regular of degree $\lambda$, but in the complement the vertices of $K$ have degree 0 while those of $S$ have degree $|S|-1$.

On the other hand, this failure of Theorem 2.2.3 is rare; it happens only for graphs of order $\lambda$. In fact, we have the following.

THEOREM 3.1.3. *Let $G \in \mathscr{G}_\kappa^\lambda$ be of order $\sigma \geqslant \lambda \geqslant \omega$. Then $G$ is regular of degree $\sigma$. Moreover, if $X \subset G$, $0 < |X| < \kappa$, then $|N(X)| = \sigma$. If $\sigma > \lambda$ then the complement $\bar{G}$ of $G$ is also regular of degree $\sigma$.*

*Proof.* The case $\sigma = \lambda$ is trivial. Suppose, therefore, that $\sigma > \lambda$. We first show that $G$ is regular. Let $X \subset G$ have cardinality less than $\kappa$. There is no loss of generality in assuming that $|(X)| = \kappa - 1$. Now each $g$ in $G - X$ is such that $N(X \cup \{g\}) \subset N(X)$ and has cardinality $\lambda$. In particular, $g \in N(Y)$ for some $\kappa$-element subset $Y$ of $N(X)$. Since $\kappa$ is finite and $|N(X)| \geqslant \lambda$ infinite, there are only $|N(X)|$ such $Y$ and each $N(Y)$ contains at most $\lambda$ elements. This means that $\sigma \leqslant \lambda \cdot |N(X)|$ and, as $\sigma > \lambda$, we must have $\sigma = |N(X)|$. This establishes the regularity of $G$.

The proof of regularity of the complement $\bar{G}$ of $G$ when $\sigma > \lambda$ is essentially that in [5]. We proceed by induction on $\kappa \geqslant 2$. Let $y \in G$ and denote by $H$ the subgraph of $G$ induced by $N(y)$. Suppose first that $\kappa = 2$. Then (by the first part of the proof) for each $x \in H$, $|N(x) \cap H| = \lambda < \sigma = |H|$. Thus $x$ has degree $\sigma$ in $\bar{H}$ and, a fortiori, in $\bar{G}$. As each vertex of $G$ lies in some $N(y)$, this completes the case of $\kappa = 2$. Assume now that $\kappa > 2$. With $y$ and $H$ as above we note that $H \in \mathscr{G}^{\lambda}_{\kappa-1}$, whence the degree of any vertex of $H$ is $\sigma$ in $\bar{H}$ and hence in $\bar{G}$. The fact that each $x$ lies in some $N(y)$ completes the proof. ∎

3.1.1. *Maximum Independent Sets in Graphs in* $\mathscr{G}^{\lambda}_{\kappa}$

For each finite $\kappa$ and each $\mu \geqslant \lambda$ we constructed graphs in $\mathscr{G}^{\lambda}_{\kappa}$ of order $\mu$ in which the cardinality of any maximum independent set is $\mu$. Is this the case for all graphs in $\mathscr{G}^{\lambda}_{\kappa}$? Clearly not: the graphs $G = K \vee S$ described just before Theorem 3.1.3 have order $\lambda$ and a maximal independent set of size $|S| < \lambda$. For $\mu > \lambda$, however, we have the following:

THEOREM 3.1.4. *Let* $G \in \mathscr{G}^{\lambda}_{\kappa}$ *be of order* $\mu > \lambda$. *Then* $G$ *contains an independent set of cardinality* $\mu$.

*Proof.* We consider two cases. First, assume that the cofinality of $\mu$ is $cf(\mu) > \lambda$. If there is no independent set of power $\mu$ then the chromatic number of $G$ is at least $\lambda^+$ and, by Theorem 3.1.1., $G$ contains a $K_{\kappa,\lambda^+}$, a contradiction.

Next, assume that $cf(\mu) \leqslant \lambda$. Fix a set $T$ of cardinality $k-1$ of vertices in $G$ and let $N = N(T)$. By Theorem 3.1.3, $|N| = \mu$. The subgraph induced by $N$ has chromatic number at most $\lambda$, as above, so there is either an independent set of power $\mu$ and we are done or a sequence of independent sets $I_\alpha$, $\alpha < \mu$, satisfying

1. $I_\alpha \subset N$
2. $|\bigcup_{\alpha < \alpha} I_\alpha| < |I_\alpha| = \mu_\alpha > \lambda$
3. $\lim_{\alpha < cf(\mu)} \mu_\alpha = \mu$.

Observe that for $u \in N$, $|N(u) \cap N| = \lambda$. Let now $I'_\alpha = I_\alpha \backslash \bigcup_{u \in J_\alpha} N(u)$, where $J_\alpha = \bigcup_{\beta < \alpha} I_\alpha$. Clearly $|\bigcup_{n \in J_\alpha} N(u) \cup N| \leqslant \lambda \cdot |J_\alpha| \leqslant |J_\alpha| < |I_\alpha|$. Thus $|I'_\alpha| = \mu_\alpha$ and $I = \bigcup_{\alpha < cf(\mu)} I'_\alpha$ is an independent set of size $\mu$. ∎

## 3.2. $\kappa$ *Infinite*

3.2.1. *Graphs of Order* $\lambda$ *in* $\mathscr{G}^{\lambda}_{\kappa}$

Recall that $\kappa < \lambda$ in this case and that every clique of order less than $\kappa^+$ can be extended to a clique of order $\kappa^+$. We give two constructions which produce $2^\lambda$ members of $\mathscr{G}_\kappa^\lambda$ of cardinality $\lambda$. First, we construct graphs of chromatic number $\lambda$.

CONSTRUCTION. Let $\Lambda$ be any graph of order at most $\lambda$ and let $G$ be the complement of the disjoint union of $\Lambda$ with an independent set of cardinality $\lambda$. Then $G \in \mathscr{G}_\kappa^\lambda$ since for any $T \subset G$ with $|T| \leqslant \kappa$ we have $|N(T)| \geqslant \lambda - |T| = \lambda$. Clearly taking non-isomorphic ordinal graphs (on $\lambda$) as $\Lambda$ will produce non-isomorphic graphs in $\mathscr{G}_\kappa^\lambda$ and hence, by Corollary 3.1.1, we have $2^\lambda$ non-isomorphic elements of $\mathscr{G}_\kappa^\lambda$. It is trivial to observe that the chromatic number of each such graph is $\lambda$ and that the complements of these graphs are not regular.

In order to construct graphs of chromatic number $\mu$, $\kappa^+ \leqslant \mu \leqslant \lambda$ we need a slightly different technique. Let $K(\mu, \lambda)$ be the complete $\mu$-partite graph with vertex partition $\{P_\alpha\}_{\alpha<\mu}$ and $|P_\alpha| = \lambda$.

THEOREM 3.2.1. *Let $\mu$ be given, $\kappa < \mu \leqslant \lambda$. Then $\mathscr{G}_\kappa^\lambda$ contains $2^\lambda$ pairwise non-isomorphic graphs of order $\lambda$ and chromatic number $\mu$.*

*Proof.* Let $\Lambda$ be an ordinal graph of order $\lambda$. Let $G_\Lambda$ be the graph obtained from $K(\mu, \lambda)$ by identifying the vertices of precisely one of the independent sets, say (without loss of generality) $P_0$, with those of $\Lambda$, thereby giving the graph induced by $P_0$, the structure of $\Lambda$. By abuse of notation we call this induced graph $\Lambda$ as well. Now $G_\Lambda \in \mathscr{G}_\kappa^\lambda$. To see this, let $T$ be a subset of $G_\Lambda$ of cardinality at most $\kappa$. Since $\kappa < \mu$, there is an $\alpha$, $0 < \alpha < \mu$, with $P_\alpha \cap T = \varnothing$. But $P_\alpha \subset N(T)$ and so $|N(T)| = \lambda$. As for the chromatic number of $G$, recall that $\Lambda$ is two-chormatic and so $G_\Lambda$ has chromatic number $\mu$. We also note that the graph thus constructed is essentially multipartite and hence is perfect.

It remains to prove that non-isomorphic ordinal graphs give non-isomorphic friendship graphs by the above construction. Let $G^0$ and $G^1$ be two graphs obtained as above from ordinal graphs $\Lambda^0$ and $\Lambda^1$, respectively. Let $P_\alpha^i$, $\alpha < \mu$, $i = 0, 1$, be the partitions of the vertex sets of $G^i$ and let, as before, $\Delta^i$ be the graph induced by $P_0^i$ in $G^i$. Let $\phi: G^0 \to G^1$ be an isomorphism. We claim that $\phi(\Lambda^0) = \Lambda^1$. Observe first that $\phi(\Lambda^0)$ cannot intersect more than two of the $P_\alpha^1$ since such an intersection would imply the existence of a triangle in a bipartite (ordinal) graph. Suppose now that $\phi(\Lambda^0)$ intersects $P_\alpha^1$ and $P_\beta^1$. Then one of the two, say $P_\alpha^1$, contains all the vertices which are of finite degree in $\phi(\Lambda^0)$. Since no vertex of $\Lambda$ is adjacent to all the vertices of finite degree in $\Lambda$, $P_\beta^1 \cap \phi(\Lambda^0) = \varnothing$. So $\phi(\Lambda^0) \subset \Lambda_0^1$. Since $\phi^{-1}$ is an isomorphism, it follows that $\phi(\Lambda^0) = \Lambda^1$. ∎

### 3.2.2. *Graphs of Order $\mu > \lambda$ in $\mathscr{G}_\kappa^\lambda$*

In the last section we constructed many examples of graphs in $\mathscr{G}_\kappa^\lambda$ of cardinality $\lambda$ and chromatic number $\mu$, where $k^+ \leqslant \mu \leqslant \lambda$. In this section we investigate the possibility that $\mathscr{G}_\kappa^\lambda$ may contain graphs of cardinality greater than $\lambda$. With the continuum hypothesis we are able to settle the question negatively. Theorems 3.2.2, 3.2.3, and 3.2.4 below establish a connection between this problem and two other combinatorial questions. With Theorem 3.2.4 it follows from Theorem 3.2.2 that there is no graph $G \in \mathscr{G}_\kappa^\lambda$ with $|G| > \lambda^\kappa$. If we assume the generalized continuum hypothesis (GCH) then Theorem 3.2.3 gives us the stronger conclusion that there is no such graph with $|G| > \lambda$.

THEOREM 3.2.2. *The existence of a graph $G$ in $\mathscr{G}_\kappa^\lambda$ with cardinality $\mu$ greater that the maximum of $2^\kappa$ and $\lambda$ entails the existence of an almost disjoint family of power $\mu$ of $\kappa$-element subsets of $\lambda$.*

*Proof.* Fix $G$ in $\mathscr{G}_\kappa^\lambda$ and assume $|G| = \mu > \lambda \cdot 2^\kappa$. Fix $T \subset G$ with $|T| = \kappa$. Now let $S = N(T)$ be the set of neighbours common to the elements of $T$ and note that $|S| = \lambda$. We now inductively choose a sequence $\{T_\alpha : \alpha < \mu\}$ of $\kappa$-element subsets of $S$ and a sequence $\{x_\alpha : \alpha < \mu\}$ of elements of $G$ so that

(i) $|T_\alpha \cap T_\beta| < \kappa$ if $\alpha \neq \beta$

(ii) $x_\alpha \notin N(T')$ for any $\kappa$-element subset $T'$ of $T_\beta$ for $\beta < \alpha$.

Suppose, therefore, that $T_\beta$ and $x_\beta$, $\beta < \alpha$, have been chosen so that (i) and (ii) are satisfied. Note that the family of those $T'$ of power $\kappa$ contained in $T_\beta$ for some $\beta < \alpha$ has at most $2^\kappa \cdot |\alpha|$ members. Hence the union of the neighbourhoods $N(T')$ has cardinality at most $2^\kappa \cdot |\alpha| \cdot \lambda < \mu$ for $\alpha < \mu$. Thus we may choose $x_\alpha$ in the complement of this union and outside $S \cup T$. Consider now $T \cup \{x_\alpha\}$. This set has power $\kappa$. Also we find $S' = N(T \cup \{x_\alpha\}) \subset N(T) = S$. Chose $T_\alpha \subset S'$ of cardinality $\kappa$. It remains to check that $|T_\alpha \cap T_\gamma| < \kappa$ for $\gamma < \alpha$. But $T' = T_\alpha \cap T_\gamma$ has power $\kappa$, contradicting the choice of $x_\alpha$, for we should have $x_\alpha \in N(T')$. This completes the proof. ∎

The existence of families of almost families of $\kappa$-subsets of $\lambda$ was investigated by Tarski [11], who showed, assuming GCH, that there is such a family $\lambda^+$ if and only if $\kappa$ and $\lambda$ have the same cofinality (see also Baumgartner [1]).

The second and apparently more fruitful translation of the problem actually gives an equivalent combinatorial problem.

Let $\kappa < \lambda$. Say that a family $\mathscr{F}$ of $\lambda$-element subsets of a fixed $\lambda$-element set $S$ is $(\kappa, \lambda)$-*friendly* if and only if

(i) for $\mathscr{F}' \subset \mathscr{F}$, if $|\mathscr{F}'| > \lambda$ then $|\bigcap \mathscr{F}'| < \kappa$, and

(ii) for $\mathscr{F}' \subset \mathscr{F}$, if $|\mathscr{F}'| \leqslant \kappa$ then $|\bigcap \mathscr{F}'| = \lambda$.

Then we have

THEOREM 3.2.3. *For $\mu \geqslant \lambda$ there is a graph $G \in \mathscr{G}_\kappa^\lambda$ of power $\mu$ just in case there is a $(\kappa, \lambda)$-friendly family $\mathscr{F}$ of cardinality $\mu$.*

*Proof.* First note that we may assume $\mu > \lambda$ (by Theorem 3.2.1 and the fact that $\langle F_\alpha \subseteq \lambda : \alpha < \lambda \rangle$, where $F_\alpha = \lambda$ is trivially $(\kappa, \lambda)$-friendly). Assume that $G \in \mathscr{G}_\kappa^\lambda$ has cardinality $\mu$. Fix a $\kappa$-element set $T$ in $G$ and its $\lambda$-element neighbourhood $N(T) = S$. Consider the family $\mathscr{F} = \{F_x : x \in G \backslash (S \cup T)\}$ of $\lambda$-subsets of $S$ where $F_x = N(T \cup \{x\})$. Note that $F_x \subset S$.

To see that (i) holds, suppose that $\mathscr{F}' \subset \mathscr{F}$ and $|\mathscr{F}'| > \lambda$. If there is a $T' \subset \bigcap \mathscr{F}'$ such that $|T'| = \kappa$ then $|N(T')| > \lambda$ since $x \in N(T')$ for each $x \in G$ such that $F_x \in \mathscr{F}'$, and this is a contradiction.

To verify (ii), assume that $\mathscr{F}' \subset \mathscr{F}$ and $|\mathscr{F}'| \leqslant \kappa$. Let $T' = \{x : F_x \in \mathscr{F}'\}$. Then $|T'| \leqslant \kappa$ and so $|T \cup T'| = \kappa$. But $N(T \cup T') \subseteq F_x$ $(x \in T')$ and so $|\bigcap \mathscr{F}'| \geqslant |N(T \cup T')| = \lambda$.

Finally, note that $F_x = F$ for at most $\lambda$ values of $x$ (otherwise $G \notin \mathscr{G}_\kappa^\lambda$), and hence $|G| \leqslant |\mathscr{F}| \cdot \lambda$. Thus $|\mathscr{F}| = \mu$.

For the converse suppose that $\mathscr{F}$ is a $(\kappa, \lambda)$-friendly family of subsets of $S$ $|\mathscr{F}| = \mu > \lambda$. Form a graph $G$ with vertex set $S \cup \{x_F : F \in \mathscr{F}\}$ and such that $S$ induces a complete subgraph, $\{x_F : F \in \mathscr{F}\}$ is an independent set, and there is an edge joining an element $s$ and a singleton $x_F$ $(F \in \mathscr{F})$ just in case $s \in F$. If $S_0 \subset S$ and $\mathscr{F}_0 \subset \mathscr{F}$ each have cardinality at most $\kappa$, then $\bigcap \mathscr{F}_0 \backslash S_0$ is a subset of $N(S_0 \cup \{x_F : F \in \mathscr{F}_0\})$, and so $|N(T)| \geqslant \lambda$ whenever $|T| \leqslant \kappa$. On the other hand, if $T \subset G$ is such that $|T| = \kappa$ and $|N(T)| > \lambda$, then $\mathscr{F}' = \{F \in \mathscr{F} : x_F \in N(T)\}$ has cardinality greater than $\lambda$ and, since $T \subset \bigcap \mathscr{F}'$, we find $|\bigcap \mathscr{F}'| \geqslant \kappa$, which contradicts (i). Hence $G \in \mathscr{G}_\kappa^\lambda$. ∎

We complete this section by giving a result which immediately gives that $\mathscr{G}_\kappa^\lambda$ contains no members of size larger than $\lambda$, assuming the GCH.

THEOREM 3.2.4. *If $\mathscr{F}$ is a $(\kappa, \lambda)$-friendly family, then $|\mathscr{F}| \leqslant \sup_{\nu < \lambda} \nu^\kappa = \underline{\lambda^\kappa}$.*

*Proof.* For a contradiction assume $|\mathscr{F}| \geqslant \mu = (\underline{\lambda^\kappa})^+$. For $F \in \mathscr{F}$ let $\alpha_F \leqslant \lambda$ be the least such that $|F \cap \alpha_F| \geqslant \kappa$. Then $\alpha_F < \lambda$, since $|F| = \lambda > \kappa$. Now for $\alpha < \lambda$ set $\mathscr{F}_\alpha = \{F \in \mathscr{F} : \alpha_F = \alpha\}$. Since $\mu$ is a regular cardinal, there is an $\alpha < \lambda$ such that $|\mathscr{F}_\alpha| \geqslant \mu$. For $S \subset \alpha$, $|S| = \kappa$, let $\mathscr{F}_s = \{F \in \mathscr{F}_\alpha : F \cap \alpha = S\}$. Again, since $\mu$ is regular and $\mu > |\alpha|^\kappa$, it follows that there is an $S \in \alpha^{(\kappa)}$ such that $|\mathscr{F}_s| \geqslant \mu$. But then $S \subset \bigcap \mathscr{F}_s$ and $|\mathscr{F}_s| \geqslant \mu > \lambda$, and this is a contradiction. ∎

*Remark.* The reader will note that we have proven the nonexistence of graphs in $\mathscr{G}_\kappa^\lambda$ of cardinality greater than $\lambda$ for any $\lambda$ such that $2^\alpha \leqslant \lambda$ for

$\alpha < \lambda$. With the GCH this implies that there are no graphs in $\mathscr{G}_\kappa^\lambda$ of power greater than $\lambda$. Note also that in the proof of Theorem 3.2.4 only the first part of the definition of a $(\kappa, \lambda)$-friendly family is used.

### 3.2.3. *Maximum Independent Sets in Graphs in* $\mathscr{G}_\kappa^\lambda$

As with $\kappa$ finite we can ask about the size of maximum independent sets in the graphs in $\mathscr{G}_\kappa^\lambda$. Here the problem becomes easier: for each $\mu < \lambda$ there is a graph in $\mathscr{G}_\kappa^\lambda$ of order $\lambda$ in which any maximal (and, hence, maximum) independent set has cardinality $\mu$. These are the complete $\lambda$-partite graphs $K(\lambda, \mu)$ with each class of the partition of cardinality $\mu$.

## 4. Comments and Questions

There are essentially two sets of questions that remain. First, in view of the remark at the end of the preceding section it makes sense to ask why the second part of the definition of a $(\kappa, \lambda)$-friendly family is not used in the proof of the last theorem while it is needed in the proof of Theorem 3.2.3.

The second series of questions stems from the GCH *not* being assumed. Is the upper bound of $\lambda$ on the order of graphs in $\mathscr{G}_\kappa^\lambda$ for $\lambda$ infinite still valid? What if we assume $\lambda$ to be regular? What happens at singular cardinals? These seem rather difficult.

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